MIDTERM EXAM: DIFFERENTIAL EQUATIONS

Duration: 3 hours **Total Marks: 20**

- Give necessary justification for all your arguments. If you are using any of the results proved in the class state them clearly.
- The functions p and q are assumed to be continuous in its domain of definition.

State true or false and EXPLAIN. Each question carries 2 marks ; Answer any <u>FOUR</u>.

1. If y is a nontrivial solution of $y'' + p(t)y' + q(t)y = 0$ in $[0, 1]$, then $\{t \in [0, 1] : y(t) = 0\}$ is a finite set.
2. The initial value problem $y'(t) = \sqrt{y}$; $y(0) = 0$ admits unique solution for $0 < t < h$, for some h.
3. Let $\alpha > 1$ and suppose there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|^{\alpha}$ for all $x, y \in [0, 1]$. Then f is a constant.
4. The functions $f(t) = t^3$ and $g(t) = t^2|t|$ are two solutions of a second order differential equation $y'' + py' + qy = 0$ where p, q are continuous functions in $[-1, 1]$.
5. If y is any solution to the differential equation $y'' + 2y' + y = 0$ in $[0, \infty)$, then $\lim_{t\to\infty} y(t) = 0$.

Answer any FOUR. Each question carries 3 marks.

1. Find the general solution of $t^2y'' - 2y = 0$ in $[1, 2]$ by guessing one solution and then using method of reduction of order find another linearly independent solution. Find the particular solution satisfying the initial conditions $y(1) = 1$ and $y'(1) = 8$. (Method of reduction of order: Look for another solution of the type $y_2(t) = C(t)y_1(t)$ when y_1 is known.)
2. Show that between any two consecutive zero's of $\sin 2t + \cos 2t$ there is precisely one zero of $\sin 2t - \cos 2t$.
3. Let $X : [0, \infty) \to I\!R^2$ and $F : [0, \infty) \times I\!R^2 \to I\!R^2$ are continuous functions with $X(0) = a \in I\!R^2$. Show that
$$\frac{dX}{dt}(t) = F(t, X(t)), \ \forall t > 0$$
if and only if
$$X(t) = a + \int_0^t F(s, X(s))ds, \ \forall t > 0.$$
4. Answer both the questions, each problem carries 1.5 marks.
   (a) If every solution to the ODE $y'' + q(t)y = 0$ is thrice differentiable, then $q(t)$ is differentiable.
   (b) Two solutions of the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ in $[\alpha, \beta]$ are linearly dependent if they have a minima or maxima at some point $t_0 \in (\alpha, \beta)$.
5. Find the general solution to $y'(t) + ay(t) = b\exp(-\lambda t)$ for $t > 0$ where λ, a, b are real numbers. Find $\lim_{t\to\infty} y(t)$ when λ, a and b are positive constants.